

An Internal Language for Categories Enriched over Generalised Metric Spaces

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Abstract

Programs with a *continuous* state space or that interact with physical processes often require notions of equivalence going beyond the standard binary setting in which equivalence either holds or does not hold. In this paper we explore the idea of equivalence taking values in a quantale \mathcal{V} , which covers the cases of (in)equations and (ultra)metric equations among others.

Our main result is the introduction of a \mathcal{V} -*equational deductive system* for linear λ -calculus together with a proof that it is sound and complete (in fact, an internal language) for a class of enriched autonomous categories. In the case of inequations, we get an internal language for autonomous categories enriched over partial orders. In the case of (ultra)metric equations, we get an internal language for autonomous categories enriched over (ultra)metric spaces.

We use our results to obtain examples of inequational and metric equational systems for higher-order programs that contain real-time and probabilistic behaviour.

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1 Introduction

Programs frequently act over a *continuous* state space or interact with physical processes like time progression or the movement of a vehicle. Such features naturally call for notions of approximation and refinement integrated in different aspects of program equivalence. Our paper falls in this line of research. Specifically, our aim is to integrate notions of approximation and refinement into the *equational system* of linear λ -calculus [4, 28, 29].

The core idea that we explore in this paper is to have equations $t =_q s$ labelled by elements q of a quantale \mathcal{V} . This covers a wide range of situations, among which the cases of (in)equations [23, 2] and metric equations [30, 31]. The latter case is perhaps less known: it consists of equations $t =_\epsilon s$ labelled by a non-negative rational number ϵ which represents the ‘maximum distance’ that the two terms t and s can be from each other. In order to illustrate metric equations, consider a programming language with a (ground) type X and a signature



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of operations $\Sigma = \{\text{wait}_n : X \rightarrow X \mid n \in \mathbb{N}\}$ that model time progression over computations of type X . Specifically, $\text{wait}_n(x)$ reads as “add a latency of n seconds to the computation x ”. In this context, the following axioms involving metric equations arise naturally:

$$\text{wait}_0(x) =_0 x \quad \text{wait}_n(\text{wait}_m(x)) =_0 \text{wait}_{n+m}(x) \quad \frac{\epsilon = |m - n|}{\text{wait}_n(x) =_\epsilon \text{wait}_m(x)} \quad (1)$$

An equation $t =_0 s$ states that the terms t and s are exactly the same and equations $t =_\epsilon s$ state that t and s differ by *at most* ϵ seconds in their execution time.

Contributions. In this paper we introduce an equational deductive system for linear λ -calculus in which equations are labelled by elements of a quantale \mathcal{V} . By using key features of a quantale’s structure, we show that this deductive system is *sound and complete* for a class of enriched symmetric monoidal closed categories (*i.e.* enriched autonomous categories). In particular, if we fix \mathcal{V} to be the Boolean quantale this class of categories consists of autonomous categories enriched over partial orders. If we fix \mathcal{V} to be the (ultra)metric quantale, this class of categories consists of autonomous categories enriched over (ultra)metric spaces. The aforementioned example of wait calls fits in the setting in which \mathcal{V} is the metric quantale. Our result provides this example with a sound and complete metric equational system, where the models are all those autonomous categories enriched over metric spaces that can soundly interpret the axioms of wait calls (1).

The next contribution of our paper falls in one of the major topics of categorical logic: to establish *logical descriptions* of certain classes of categories. A famous result of this kind is the correspondence between λ -calculus and Cartesian closed categories which states that the former is the *internal language* of the latter [24] – such a correspondence allows to study Cartesian closed categories by means of logical tools. An analogous result is presented in [28, 29] for linear λ -calculus and symmetric monoidal closed (*i.e.* autonomous) categories. We show that linear λ -calculus equipped with a \mathcal{V} -equational system is the internal language of autonomous categories enriched over ‘generalised metric spaces’.

Outline. Section 2 recalls linear λ -calculus, its equational system, and the famous correspondence to autonomous categories, via soundness, completeness, and internal language theorems. The contents of this section are slight adaptations of results presented in [28, 4], the main difference being that we forbid the exchange rule to be explicitly part of linear λ -calculus (instead it is only admissible). This choice is important to ensure that judgements in the calculus have *unique* derivations, which allows us to refer to their interpretations unambiguously [37]. Section 3 presents the main contributions of this paper. It walks a path analogous to Section 2, but now in the setting of \mathcal{V} -equations (*i.e.* equations labelled by elements of a quantale \mathcal{V}). As we will see, the semantic counterpart of moving from equations to \mathcal{V} -equations is to move from categories to categories enriched over \mathcal{V} -categories. The latter, often regarded as generalised metric spaces, are central entities in a fruitful area of enriched category theory that aims to treat uniformly different kinds of “structured sets”, such as partial orders, fuzzy partial orders, and (ultra)metric spaces [25, 38, 39]. Our results are applicable to all these examples. Section 4 presents some examples of \mathcal{V} -equational axioms and corresponding models. Specifically, we will revisit the axioms of wait calls (1) and consider an inequational variant. Then we will study a metric axiom for probabilistic programs and show that the category of Banach spaces and short linear maps is a model for the resulting metric theory. We will additionally use this example to illustrate how our deductive system allows to compute an approximate distance between two probabilistic

programs easily as opposed to computing an exact distance “semantically” which tends to involve quite complex operators. Finally, Section 5 establishes a functorial connection between our results and previous well-known semantics for linear logic [11, 29], and concludes with a brief exposition of future work. We assume knowledge of λ -calculus and category theory [28, 29, 24, 27]. Proofs omitted in the text are available in the extended version of this paper [10].

Related work. Several approaches to incorporating quantitative information to programming languages have been explored in the literature. Closest to this work are various approaches targeted at λ -calculi. In [7, 8] a notion of distance called *context distance* is developed, first for an affine, then for a more general λ -calculus, with probabilistic programs as the main motivation. [14] considers a notion of quantale-valued *applicative (bi)similarity*, an operational *coinductive* technique used for showing contextual equivalence between two programs. Recently, [34] presented several Cartesian closed categories of generalised metric spaces that provide a quantitative semantics to simply-typed λ -calculus based on a generalisation of logical relations. None of these examples reason about distances in a quantitative equational system, and in this respect our work is closer to the metric universal algebra developed in [30, 31].

A different approach consists in encoding quantitative information via a type system. In particular, graded (modal) types [16, 13, 32] have found applications in *e.g.* differential privacy [35] and information flow [1]. This approach is to some extent orthogonal to ours as it mainly aims to model coeffects, whilst we aim to reason about the intrinsic quantitative nature of λ -terms acting *e.g.* on continuous or ordered spaces.

Quantum programs provide an interesting example of intrinsically quantitative programs, by which we mean that the metric structure on quantum states does not arise from (co)effects. Recently, [19] showed how the issue of noise in a quantum while-language can be handled by developing a *deductive system* to determine how similar a quantum program is from its idealised, noise-free version; an approach very much in the spirit of this work.

2 An internal language for autonomous categories

In this section we briefly recall linear λ -calculus, which can be regarded as a term assignment system for the exponential free, multiplicative fragment of *intuitionistic linear logic*. Then we recall that it is sound and complete w.r.t. autonomous categories, and also that it is an internal language for such categories. We mention only what is needed to present our results, the interested reader will find a more detailed exposition in [28, 4, 29]. Let us start by fixing a *class* G of ground types. The grammar of types for linear λ -calculus is given by:

$$\mathbb{A} ::= X \in G \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A}$$

We also fix a class Σ of sorted operation symbols $f : \mathbb{A}_1, \dots, \mathbb{A}_n \rightarrow \mathbb{A}$ with $n \geq 1$. As usual, we use Greek letters Γ, Δ, E, \dots to denote *typing contexts*, *i.e.* lists $x_1 : \mathbb{A}_1, \dots, x_n : \mathbb{A}_n$ of typed variables such that each variable x_i occurs at most once in x_1, \dots, x_n .

We will use the notion of a shuffle for building a linear typing system such that the *exchange rule* is admissible and each judgement $\Gamma \triangleright v : \mathbb{A}$ (details about these below) has a *unique derivation* – this will allow us to refer to a judgement’s denotation $\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket$ unambiguously. By shuffle we mean a permutation of typed variables in a context sequence $\Gamma_1, \dots, \Gamma_n$ such that for all $i \leq n$ the relative order of the variables in Γ_i is preserved [37]. For example, if $\Gamma_1 = x : \mathbb{A}, y : \mathbb{B}$ and $\Gamma_2 = z : \mathbb{C}$ then $z : \mathbb{C}, x : \mathbb{A}, y : \mathbb{B}$ is a shuffle but

$y : \mathbb{B}, x : \mathbb{A}, z : \mathbb{C}$ is *not*, because we changed the order in which x and y appear in Γ_1 . As explained in [37] (and also in the proof of Lemma 1), such a restriction on relative orders is crucial for judgements having unique derivations. We denote by $\text{Sf}(\Gamma_1; \dots; \Gamma_n)$ the set of shuffles on $\Gamma_1, \dots, \Gamma_n$.

The term formation rules of linear λ -calculus are listed in Fig. 1. They correspond to the natural deduction rules of the exponential free, multiplicative fragment of intuitionistic linear logic.

► **Lemma 1.** *All judgements $\Gamma \triangleright v : \mathbb{A}$ have a unique derivation.*

Substitution is defined in the expected way, and the following result is standard.

► **Lemma 2 (Exchange and Substitution).** *For every judgement $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \triangleright v : \mathbb{C}$ we can derive $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \triangleright v : \mathbb{C}$. For all judgements $\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}$ and $\Delta \triangleright w : \mathbb{A}$ we can derive $\Gamma, \Delta \triangleright v[w/x] : \mathbb{B}$.*

We now recall the interpretation of judgements $\Gamma \triangleright v : \mathbb{A}$ in a symmetric monoidal closed (autonomous) category \mathbf{C} . But before proceeding with this description, let us fix notation for some of the constructions available in autonomous categories. For all \mathbf{C} -objects X, Y, Z , $\text{sw} : X \otimes Y \rightarrow Y \otimes X$ denotes the symmetry morphism, $\lambda : \mathbb{I} \otimes X \rightarrow X$ the left unitor, $\text{app} : (X \multimap Y) \otimes X \rightarrow Y$ the application, and $\alpha : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$ the left associator. Moreover for all \mathbf{C} -morphisms $f : X \otimes Y \rightarrow Z$ we denote the corresponding carried version (right transpose) by $\bar{f} : X \rightarrow (Y \multimap Z)$.

For all ground types $X \in G$ we postulate an interpretation $\llbracket X \rrbracket$ as a \mathbf{C} -object. Types are then interpreted by induction over the type structure of linear λ -calculus, using the tensor \otimes and exponential \multimap constructs of autonomous categories. Given a non-empty context $\Gamma = \Gamma', x : \mathbb{A}$, its interpretation is defined by $\llbracket \Gamma', x : \mathbb{A} \rrbracket = \llbracket \Gamma' \rrbracket \otimes \llbracket \mathbb{A} \rrbracket$ if Γ' is non-empty and $\llbracket \Gamma', x : \mathbb{A} \rrbracket = \llbracket \mathbb{A} \rrbracket$ otherwise. The empty context $-$ is interpreted as $\llbracket - \rrbracket = \mathbb{I}$ where \mathbb{I} is the unit of \otimes in \mathbf{C} . To keep notation simple, given $X_1, \dots, X_n \in \mathbf{C}$ we write $X_1 \otimes \dots \otimes X_n$ for the n -tensor $(\dots (X_1 \otimes X_2) \otimes \dots) \otimes X_n$, and similarly for \mathbf{C} -morphisms.

We will also need some ‘housekeeping’ morphisms to handle interactions between context interpretation and the autonomous structure of \mathbf{C} . Specifically, given contexts $\Gamma_1, \dots, \Gamma_n$ we denote by $\text{sp}_{\Gamma_1; \dots; \Gamma_n} : \llbracket \Gamma_1, \dots, \Gamma_n \rrbracket \rightarrow \llbracket \Gamma_1 \rrbracket \otimes \dots \otimes \llbracket \Gamma_n \rrbracket$ the morphism that splits $\llbracket \Gamma_1, \dots, \Gamma_n \rrbracket$ into $\llbracket \Gamma_1 \rrbracket \otimes \dots \otimes \llbracket \Gamma_n \rrbracket$, and by $\text{jn}_{\Gamma_1; \dots; \Gamma_n}$ the corresponding inverse. Given a context $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta$ we denote by $\text{exch}_{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta} : \llbracket \Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rrbracket \rightarrow \llbracket \Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rrbracket$ the morphism corresponding to the permutation of the variable $x : \mathbb{A}$ with $y : \mathbb{B}$. Whenever convenient we will drop variable names in the subscripts of sp , jn , and exch . For a context $E \in \text{Sf}(\Gamma_1, \dots, \Gamma_n)$ the morphism $\text{sh}_E : \llbracket E \rrbracket \rightarrow \llbracket \Gamma_1, \dots, \Gamma_n \rrbracket$ denotes the corresponding shuffling morphism.

For every operation symbol $f : \mathbb{A}_1, \dots, \mathbb{A}_n \rightarrow \mathbb{A}$ in Σ we postulate an interpretation $\llbracket f \rrbracket : \llbracket \mathbb{A}_1 \rrbracket \otimes \dots \otimes \llbracket \mathbb{A}_n \rrbracket \rightarrow \llbracket \mathbb{A} \rrbracket$ as a \mathbf{C} -morphism. The interpretation of judgements is defined by induction over the structure of judgement derivation according to the rules in Fig. 2.

As detailed in [4, 28, 29], linear λ -calculus comes equipped with a class of equations (Fig. 3), specifically *equations-in-context* $\Gamma \triangleright v = w : \mathbb{A}$, that corresponds to the axiomatics of autonomous categories. As usual, we omit the context and typing information of the equations in Fig. 3, which can be reconstructed in the usual way.

► **Theorem 3.** *The equations presented in Fig. 3 are sound w.r.t. judgement interpretation. Specifically if $\Gamma \triangleright v = w : \mathbb{A}$ is one of the equations in Fig. 3 then $\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket = \llbracket \Gamma \triangleright w : \mathbb{A} \rrbracket$.*

► **Definition 4 (Linear λ -theories).** *Consider a tuple (G, Σ) consisting of a class G of ground types and a class Σ of sorted operation symbols. A linear λ -theory $((G, \Sigma), Ax)$ is a triple such that Ax is a class of equations-in-context over linear λ -terms built from (G, Σ) .*

$$\begin{array}{c}
\frac{\Gamma_i \triangleright v_i : \mathbb{A}_i \quad f : \mathbb{A}_1, \dots, \mathbb{A}_n \rightarrow \mathbb{A} \in \Sigma \quad E \in \text{Sf}(\Gamma_1; \dots; \Gamma_n)}{E \triangleright f(v_1, \dots, v_n) : \mathbb{A}} \text{ (ax)} \quad \frac{}{x : \mathbb{A} \triangleright x : \mathbb{A}} \text{ (hyp)} \\
\\
\frac{}{- \triangleright * : \mathbb{I}} \text{ (I}_i\text{)} \quad \frac{\Gamma \triangleright v : \mathbb{I} \quad \Delta \triangleright w : \mathbb{A} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright v \text{ to } * . w : \mathbb{A}} \text{ (I}_e\text{)} \\
\\
\frac{\Gamma \triangleright v : \mathbb{A} \quad \Delta \triangleright w : \mathbb{B} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright v \otimes w : \mathbb{A} \otimes \mathbb{B}} \text{ (\otimes}_i\text{)} \\
\\
\frac{\Gamma \triangleright v : \mathbb{A} \otimes \mathbb{B} \quad \Delta, x : \mathbb{A}, y : \mathbb{B} \triangleright w : \mathbb{C} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright \text{pm } v \text{ to } x \otimes y . w : \mathbb{C}} \text{ (\otimes}_e\text{)} \\
\\
\frac{\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}}{\Gamma \triangleright \lambda x : \mathbb{A} . v : \mathbb{A} \multimap \mathbb{B}} \text{ (-}\circ\text{i)} \quad \frac{\Gamma \triangleright v : \mathbb{A} \multimap \mathbb{B} \quad \Delta \triangleright w : \mathbb{A} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright v w : \mathbb{B}} \text{ (-}\circ\text{e)}
\end{array}$$

■ **Figure 1** Term formation rules for linear λ -calculus.

$$\begin{array}{c}
\frac{[\Gamma_i \triangleright v_i : \mathbb{A}_i] = m_i \quad f : \mathbb{A}_1, \dots, \mathbb{A}_n \rightarrow \mathbb{A} \in \Sigma \quad E \in \text{Sf}(\Gamma_1 \dots \Gamma_n)}{[E \triangleright f(v_1, \dots, v_n) : \mathbb{A}] = [f] \cdot (m_1 \otimes \dots \otimes m_n) \cdot \text{sp}_{\Gamma_1, \dots, \Gamma_n} \cdot \text{sh}_E} \quad [x : \mathbb{A} \triangleright x : \mathbb{A}] = \text{id}_{[\mathbb{A}]} \\
\\
\frac{}{[- \triangleright * : \mathbb{I}] = \text{id}_{[\mathbb{I}]}} \quad \frac{[\Gamma \triangleright v : \mathbb{I}] = m \quad [\Delta \triangleright w : \mathbb{A}] = n \quad E \in \text{Sf}(\Gamma; \Delta)}{[E \triangleright v \text{ to } * . w : \mathbb{A}] = n \cdot \lambda \cdot (m \otimes \text{id}) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E} \\
\\
\frac{[\Gamma \triangleright v : \mathbb{A}] = m \quad [\Delta \triangleright w : \mathbb{B}] = n \quad E \in \text{Sf}(\Gamma; \Delta)}{[E \triangleright v \otimes w : \mathbb{A} \otimes \mathbb{B}] = (m \otimes n) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E} \quad \frac{[\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}] = m}{[\Gamma \triangleright \lambda x : \mathbb{A} . v : \mathbb{A} \multimap \mathbb{B}] = (m \cdot \text{jn}_{\Gamma; \mathbb{A}})} \\
\\
\frac{[\Gamma \triangleright v : \mathbb{A} \otimes \mathbb{B}] = m \quad [\Delta, x : \mathbb{A}, y : \mathbb{B} \triangleright w : \mathbb{C}] = n \quad E \in \text{Sf}(\Gamma; \Delta)}{[E \triangleright \text{pm } v \text{ to } x \otimes y . w : \mathbb{C}] = n \cdot \text{jn}_{\Delta; \mathbb{A}; \mathbb{B}} \cdot \alpha \cdot \text{sw} \cdot (m \otimes \text{id}) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E} \\
\\
\frac{[\Gamma \triangleright v : \mathbb{A} \multimap \mathbb{B}] = m \quad [\Delta \triangleright w : \mathbb{A}] = n \quad E \in \text{Sf}(\Gamma; \Delta)}{[E \triangleright v w : \mathbb{B}] = \text{app} \cdot (m \otimes n) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E}
\end{array}$$

■ **Figure 2** Judgement interpretation on an autonomous category \mathcal{C} .

$$\begin{array}{ll}
\text{pm } v \otimes w \text{ to } x \otimes y . u & = u[v/x, w/y] \\
\text{pm } v \text{ to } x \otimes y . u[x \otimes y/z] & = u[v/z] \\
* \text{ to } * . v & = v \\
v \text{ to } * . w[* / z] & = w[v/z]
\end{array}
\quad
\begin{array}{ll}
(\lambda x : \mathbb{A} . v) w & = v[w/x] \\
\lambda x : \mathbb{A} . (v x) & = v
\end{array}$$

(a) Monoidal structure.

(b) Higher-order structure.

$$\begin{array}{ll}
u[v \text{ to } * . w/z] & = v \text{ to } * . u[w/z] \\
u[\text{pm } v \text{ to } x \otimes y . w/z] & = \text{pm } v \text{ to } x \otimes y . u[w/z]
\end{array}$$

(c) Commuting conversions.

■ **Figure 3** Equations corresponding to the axiomatics of autonomous categories.

The elements of Ax are called axioms (of the theory). Let $Th(Ax)$ be the smallest congruence that contains Ax , the equations listed in Fig. 3, and that is closed under the exchange and substitution rules. We call the elements of $Th(Ax)$ theorems (of the theory).

► **Definition 5** (Models of linear λ -theories). *Consider a linear λ -theory $((G, \Sigma), Ax)$ and an autonomous category \mathcal{C} . Suppose that for each $X \in G$ we have an interpretation $\llbracket X \rrbracket$ that is a \mathcal{C} -object and analogously for the operation symbols. This interpretation structure is a model of the theory if all axioms are satisfied by the interpretation.*

Next let us turn our attention to the correspondence between linear λ -calculus and autonomous categories, established via soundness, completeness, and internal language theorems. Despite the proofs of such theorems already being detailed in [28, 4, 29], we decided to briefly sketch them below to render the presentation of some of our own results self-contained.

► **Theorem 6** (Soundness & Completeness). *Consider a linear λ -theory T . An equation $\Gamma \triangleright v = w : \mathbb{A}$ is a theorem of T iff it is satisfied by all models of the theory.*

Proof sketch. Soundness follows by induction over the rules that define $Th(Ax)$ (Definition 4) and by Theorem 3. Completeness is based on the idea of a Lindenbaum-Tarski algebra: it follows from building the *syntactic category* $\text{Syn}(T)$ of T (also known as term model), showing that it possesses an autonomous structure and also that equality $\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket = \llbracket \Gamma \triangleright w : \mathbb{A} \rrbracket$ in the syntactic category is equivalent to provability $\Gamma \triangleright v = w : \mathbb{A}$ in the theory.

The syntactic category of T has as objects the types of T and as morphisms $\mathbb{A} \rightarrow \mathbb{B}$ the equivalence classes (w.r.t. provability) of terms v for which we can derive $x : \mathbb{A} \triangleright v : \mathbb{B}$. ◀

Next let us focus on the topic of internal languages, for which the following result is quite useful.

► **Theorem 7.** *Consider a linear λ -theory T and a model of T on an autonomous category \mathcal{C} . The model induces a functor $F : \text{Syn}(T) \rightarrow \mathcal{C}$ that (strictly) preserves the autonomous structure.*

Proof sketch. Consider a model of T on a category \mathcal{C} . Then for any judgement $x : \mathbb{A} \triangleright v : \mathbb{B}$, the induced functor F sends the equivalence class $[v]$ into $\llbracket x : \mathbb{A} \triangleright v : \mathbb{B} \rrbracket$. ◀

An autonomous category \mathcal{C} induces a linear λ -theory $\text{Lang}(\mathcal{C})$ whose ground types $X \in G$ are the objects of \mathcal{C} and whose signature Σ of operation symbols consists of all the morphisms in \mathcal{C} plus certain isomorphisms that we describe in (2). The axioms of $\text{Lang}(\mathcal{C})$ are all the equations satisfied by the obvious interpretation in \mathcal{C} . In order to explicitly distinguish the autonomous structure of \mathcal{C} from the type structure of $\text{Lang}(\mathcal{C})$ let us denote the tensor of \mathcal{C} by $\hat{\otimes}$, the unit by $\hat{\mathbb{I}}$, and the exponential by $\hat{\multimap}$. Consider then the following map on types:

$$i(\mathbb{I}) = \hat{\mathbb{I}} \quad i(X) = X \quad i(\mathbb{A} \otimes \mathbb{B}) = i(\mathbb{A}) \hat{\otimes} i(\mathbb{B}) \quad i(\mathbb{A} \multimap \mathbb{B}) = i(\mathbb{A}) \hat{\multimap} i(\mathbb{B}) \quad (2)$$

For each type \mathbb{A} we add an isomorphism $\mathbb{A} \simeq i(\mathbb{A})$ to the theory $\text{Lang}(\mathcal{C})$.

► **Theorem 8** (Internal language). *For every autonomous category \mathcal{C} there exists an equivalence of categories $\text{Syn}(\text{Lang}(\mathcal{C})) \simeq \mathcal{C}$.*

Proof sketch. By construction, we have an interpretation of $\text{Lang}(\mathcal{C})$ in \mathcal{C} which behaves as the identity for operation symbols and ground types. This interpretation is a model of $\text{Lang}(\mathcal{C})$ on \mathcal{C} and by Theorem 7 we obtain a functor $\text{Syn}(\text{Lang}(\mathcal{C})) \rightarrow \mathcal{C}$. The functor in the opposite direction behaves as the identity on objects and sends a \mathcal{C} -morphism f into $\llbracket f(x) \rrbracket$. The equivalence of categories is then shown by using the aforementioned isomorphisms which connect the type constructors of $\text{Lang}(\mathcal{C})$ with the autonomous structure of \mathcal{C} . ◀

3 From equations to \mathcal{V} -equations

We now extend the results of the previous section to the setting of \mathcal{V} -equations.

3.1 A \mathcal{V} -equational deductive system

Let \mathcal{V} denote a commutative and unital quantale, $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ the corresponding binary operation, and k the corresponding unit [33]. As mentioned in the introduction, \mathcal{V} induces the notion of a \mathcal{V} -equation, *i.e.* an equation $t =_q s$ labelled by an element q of \mathcal{V} . This subsection explores this concept by introducing a \mathcal{V} -equational deductive system for linear λ -calculus and a notion of a linear $\mathcal{V}\lambda$ -theory.

Let us start by recalling two definitions concerning ordered structures [15, 17] and then explain their relevance to our work.

► **Definition 9.** Consider a complete lattice L . For every $x, y \in L$ we say that y is way-below x (in symbols, $y \ll x$) if for every subset $X \subseteq L$ whenever $x \leq \bigvee X$ there exists a finite subset $A \subseteq X$ such that $y \leq \bigvee A$. The lattice L is called continuous iff for every $x \in L$,

$$x = \bigvee \{y \mid y \in L \text{ and } y \ll x\}$$

► **Definition 10.** Let L be a complete lattice. A basis B of L is a subset $B \subseteq L$ such that for every $x \in L$ the set $B \cap \{y \mid y \in L \text{ and } y \ll x\}$ is directed and has x as the least upper bound.

From now on we assume that the underlying lattice of \mathcal{V} is continuous and has a basis B which is closed under finite joins, the multiplication of the quantale \otimes and contains the unit k . These assumptions will allow us to work *only* with a specified subset of \mathcal{V} -equations chosen *e.g.* for computational reasons, such as the *finite* representation of values $q \in \mathcal{V}$.

► **Example 11.** The Boolean quantale $((\{0 \leq 1\}, \vee), \otimes := \wedge)$ is *finite* and thus continuous [15]. Since it is continuous, $\{0, 1\}$ itself is a basis for the quantale that satisfies the conditions above. For the Gödel t-norm [12] $(([0, 1], \vee), \otimes := \wedge)$, the way-below relation is the strictly-less relation $<$ with the exception that $0 < 0$. A basis for the underlying lattice that satisfies the conditions above is the set $\mathbb{Q} \cap [0, 1]$. Note that, unlike real numbers, rational numbers always have a finite representation. For the metric quantale (also known as Lawvere quantale) $(([0, \infty], \wedge), \otimes := +)$, the way-below relation corresponds to the *strictly greater* relation with $\infty > \infty$, and a basis for the underlying lattice that satisfies the conditions above is the set of extended non-negative rational numbers. The latter also serves as basis for the ultrametric quantale $(([0, \infty], \wedge), \otimes := \max)$.

We also assume that \mathcal{V} is *integral*, *i.e.* that the unit k is the top element of \mathcal{V} . This will allow us to establish a smoother theory of \mathcal{V} -equations, whilst still covering *e.g.* all the examples above. This assumption is common in quantale theory [39].

Recall the term formation rules of linear λ -calculus from Fig. 1. A \mathcal{V} -equation-in-context is an expression $\Gamma \triangleright v =_q w : \mathbb{A}$ with $q \in B$ (the basis of \mathcal{V}), $\Gamma \triangleright v : \mathbb{A}$ and $\Gamma \triangleright w : \mathbb{A}$. Let \top be the top element in \mathcal{V} . An equation-in-context $\Gamma \triangleright v = w : \mathbb{A}$ now denotes the particular case in which both $\Gamma \triangleright v =_{\top} w : \mathbb{A}$ and $\Gamma \triangleright w =_{\top} v : \mathbb{A}$. For the case of the Boolean quantale, \mathcal{V} -equations are labelled by $\{0, 1\}$. We will see that $\Gamma \triangleright v =_1 w : \mathbb{A}$ can be treated as an inequation $\Gamma \triangleright v \leq w : \mathbb{A}$, whilst $\Gamma \triangleright v =_0 w : \mathbb{A}$ corresponds to a trivial \mathcal{V} -equation, *i.e.* a \mathcal{V} -equation that always holds. For the Gödel t-norm, we can choose $\mathbb{Q} \cap [0, 1]$ as basis and then obtain what we call *fuzzy inequations*. For the metric quantale, we can choose the set of extended non-negative rational numbers as basis and then obtain *metric equations* in

the spirit of [30, 31]. Similarly, by choosing the ultrametric quantale $(([0, \infty], \wedge), \otimes := \max)$ with the set of extended non-negative rational numbers as basis we obtain what we call *ultrametric equations*.

► **Definition 12** (Linear $\mathcal{V}\lambda$ -theories). *Consider a tuple (G, Σ) consisting of a class G of ground types and a class of sorted operation symbols $f : \mathbb{A}_1, \dots, \mathbb{A}_n \rightarrow \mathbb{A}$ with $n \geq 1$. A linear $\mathcal{V}\lambda$ -theory $((G, \Sigma), Ax)$ is a tuple such that Ax is a class of \mathcal{V} -equations-in-context over linear λ -terms built from (G, Σ) .*

$$\begin{array}{c}
 \frac{}{v =_{\top} v} \text{ (refl)} \qquad \frac{v =_q w \quad w =_r u}{v =_{q \otimes r} u} \text{ (trans)} \qquad \frac{v =_q w \quad r \leq q}{v =_r w} \text{ (weak)} \\
 \\
 \frac{\forall r \ll q. v =_r w}{v =_q w} \text{ (arch)} \qquad \frac{\forall i \leq n. v =_{q_i} w}{v =_{\vee q_i} w} \text{ (join)} \\
 \hline
 \frac{\forall i \leq n. v_i =_{q_i} w_i}{f(v_1, \dots, v_n) =_{\otimes q_i} f(w_1, \dots, w_n)} \qquad \frac{v =_q w \quad v' =_r w'}{v \otimes v' =_{q \otimes r} w \otimes w'} \\
 \\
 \frac{v =_q w \quad v' =_r w'}{\text{pm } v \text{ to } x \otimes y. v' =_{q \otimes r} \text{pm } w \text{ to } x \otimes y. w'} \\
 \\
 \frac{v =_q w \quad v' =_r w'}{v \text{ to } *. v' =_{q \otimes r} w \text{ to } *. w'} \qquad \frac{v =_q w}{\lambda x : \mathbb{A}. v =_q \lambda x : \mathbb{A}. w} \qquad \frac{v =_q w \quad v' =_r w'}{v v' =_{q \otimes r} w w'} \\
 \\
 \frac{\Gamma \triangleright v =_q w : \mathbb{A} \quad \Delta \in \text{perm}(\Gamma)}{\Delta \triangleright v =_q w : \mathbb{A}} \qquad \frac{v =_q w \quad v' =_r w'}{v[v'/x] =_{q \otimes r} w[w'/x]}
 \end{array}$$

■ **Figure 4** \mathcal{V} -congruence rules.

The elements of Ax are the axioms of the theory. Let $Th(Ax)$ be the smallest class that contains Ax and that is closed under the rules of Fig. 3 and of Fig. 4 (as usual we omit the context and typing information). The elements of $Th(Ax)$ are the theorems of the theory.

Let us examine the rules in Fig. 4 in more detail. They can be seen as a generalisation of the notion of a congruence. The rules **(refl)** and **(trans)** are a generalisation of equality's reflexivity and transitivity. Rule **(weak)** encodes the principle that the higher the label in the \mathcal{V} -equation, the “tighter” is the relation between the two terms in the \mathcal{V} -equation. In other words, $v =_r w$ is subsumed by $v =_q w$, for $r \leq q$. This can be seen clearly *e.g.* with the metric quantale by reading $v =_q w$ as “the terms v and w are *at most* at distance q from each other” (recall that in the metric quantale the usual order is reversed, *i.e.* $\leq := \geq_{[0, \infty]}$). **(arch)** is essentially a generalisation of the Archimedean rule in [30, 31]. It says that if $v =_r w$ for all *approximations* r of q then it is also the case that $v =_q w$. **(join)** says that deductions are closed under finite joins, and in particular it is always the case that $v =_{\perp} w$. All other rules correspond to a generalisation of *compatibility* to a \mathcal{V} -equational setting.

The reader may have noticed that the rules in Fig. 4 do not contain a \mathcal{V} -generalisation of symmetry w.r.t. standard equality. Such a generalisation would be:

$$\frac{v =_q w}{w =_q v}$$

This rule is not present in Fig. 4 because in some quantales \mathcal{V} it forces too many \mathcal{V} -equations. For example, in the Boolean quantale the condition $v \leq w$ would automatically entail $w \leq v$

(due to symmetry); in fact, for this particular case symmetry forces the notion of inequation to collapse into the classical notion of equation. On the other hand, symmetry is desirable in the (ultra)metric case because (ultra)metrics need to respect the symmetry equation [17].

► **Definition 13** (Symmetric linear $\mathcal{V}\lambda$ -theories). *A symmetric linear $\mathcal{V}\lambda$ -theory is a linear $\mathcal{V}\lambda$ -theory whose set of theorems is closed under symmetry.*

In the paper’s extended version [10] we further explore how specific families of quantales are reflected in the \mathcal{V} -equational system here introduced, and briefly compare the latter to metric algebra [30, 31].

3.2 Semantics of \mathcal{V} -equations

In this subsection we set the necessary background for presenting a sound and complete class of models for (symmetric) linear $\mathcal{V}\lambda$ -theories. We start by recalling basic concepts of \mathcal{V} -categories, which are central in a field initiated by Lawvere in [25] and can be intuitively seen as generalised metric spaces [38, 18, 39]. As we will see, \mathcal{V} -categories provide structure to suitably interpret \mathcal{V} -equations.

► **Definition 14.** *A (small) \mathcal{V} -category is a pair (X, a) where X is a class (set) and $a : X \times X \rightarrow \mathcal{V}$ is a function that satisfies:*

$$k \leq a(x, x) \quad \text{and} \quad a(x, y) \otimes a(y, z) \leq a(x, z) \quad (x, y, z \in X)$$

For two \mathcal{V} -categories (X, a) and (Y, b) , a \mathcal{V} -functor $f : (X, a) \rightarrow (Y, b)$ is a function $f : X \rightarrow Y$ that satisfies the inequality $a(x, y) \leq b(f(x), f(y))$ for all $x, y \in X$.

Small \mathcal{V} -categories and \mathcal{V} -functors form a category which we denote by $\mathcal{V}\text{-Cat}$. A \mathcal{V} -category (X, a) is called *symmetric* if $a(x, y) = a(y, x)$ for all $x, y \in X$. We denote by $\mathcal{V}\text{-Cat}_{\text{sym}}$ the full subcategory of $\mathcal{V}\text{-Cat}$ whose objects are symmetric. Every \mathcal{V} -category carries a natural order defined by $x \leq y$ whenever $k \leq a(x, y)$. A \mathcal{V} -category is called *separated* if its natural order is anti-symmetric. We denote by $\mathcal{V}\text{-Cat}_{\text{sep}}$ the full subcategory of $\mathcal{V}\text{-Cat}$ whose objects are separated.

► **Example 15.** For \mathcal{V} the Boolean quantale, $\mathcal{V}\text{-Cat}_{\text{sep}}$ is the category Pos of partially ordered sets and monotone maps; $\mathcal{V}\text{-Cat}_{\text{sym, sep}}$ is simply the category Set of sets and functions. For \mathcal{V} the metric quantale, $\mathcal{V}\text{-Cat}_{\text{sym, sep}}$ is the category Met of extended metric spaces and non-expansive maps. In what follows we omit the qualifier “extended” in “extended (ultra)metric spaces”. For \mathcal{V} the ultrametric quantale, $\mathcal{V}\text{-Cat}_{\text{sym, sep}}$ is the category of ultrametric spaces and non-expansive maps.

The inclusion functor $\mathcal{V}\text{-Cat}_{\text{sep}} \hookrightarrow \mathcal{V}\text{-Cat}$ has a left adjoint [18]. It is constructed first by defining the equivalence relation $x \sim y$ whenever $x \leq y$ and $y \leq x$ (for \leq the natural order introduced earlier). Then this relation induces the separated \mathcal{V} -category $(X/\sim, \tilde{a})$ where \tilde{a} is defined as $\tilde{a}([x], [y]) = a(x, y)$ for every $[x], [y] \in X/\sim$. The left adjoint of the inclusion functor $\mathcal{V}\text{-Cat}_{\text{sep}} \hookrightarrow \mathcal{V}\text{-Cat}$ sends every \mathcal{V} -category (X, a) to $(X/\sim, \tilde{a})$. This quotienting construct preserves symmetry, and therefore we automatically obtain the following result.

► **Theorem 16.** *The inclusion functor $\mathcal{V}\text{-Cat}_{\text{sym, sep}} \hookrightarrow \mathcal{V}\text{-Cat}_{\text{sym}}$ has a left adjoint.*

Next, we recall notions of enriched category theory [20] instantiated into the setting of *autonomous categories enriched over \mathcal{V} -categories*. We will use the enriched structure to give semantics to \mathcal{V} -equations between linear λ -terms. First, note that every category $\mathcal{V}\text{-Cat}$

is autonomous with the tensor $(X, a) \otimes (Y, b) := (X \times Y, a \otimes b)$ where $a \otimes b$ is defined as $(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y')$ and the set of \mathcal{V} -functors $\mathcal{V}\text{-Cat}((X, a), (Y, b))$ equipped with the map $(f, g) \mapsto \bigwedge_{x \in X} b(f(x), g(x))$.

► **Theorem 17.** *The categories $\mathcal{V}\text{-Cat}_{\text{sym}}$, $\mathcal{V}\text{-Cat}_{\text{sep}}$, and $\mathcal{V}\text{-Cat}_{\text{sym, sep}}$ inherit the autonomous structure of $\mathcal{V}\text{-Cat}$ whenever \mathcal{V} is integral.*

Since we assume that \mathcal{V} is integral, this last theorem allows us to formally define the notion of categories enriched over \mathcal{V} -categories using [20].

► **Definition 18.** *A category \mathcal{C} is $\mathcal{V}\text{-Cat}$ -enriched (or simply, a $\mathcal{V}\text{-Cat}$ -category) if for all \mathcal{C} -objects X and Y the hom-set $\mathcal{C}(X, Y)$ is a \mathcal{V} -category and if the composition of \mathcal{C} -morphisms,*

$$(\cdot) : \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z)$$

is a \mathcal{V} -functor. Given two $\mathcal{V}\text{-Cat}$ -categories \mathcal{C} and \mathcal{D} and a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we call F a $\mathcal{V}\text{-Cat}$ -functor if for all \mathcal{C} -objects X and Y the map $F_{X, Y} : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$ is a \mathcal{V} -functor. An adjunction $\mathcal{C} : F \dashv G : \mathcal{D}$ is called $\mathcal{V}\text{-Cat}$ -enriched if the underlying functors F and G are $\mathcal{V}\text{-Cat}$ -functors and if for all objects $X \in |\mathcal{C}|$ and $Y \in |\mathcal{D}|$ there exists a \mathcal{V} -isomorphism $\mathcal{D}(FX, Y) \simeq \mathcal{C}(X, GY)$ natural in X and Y .

If \mathcal{C} is a $\mathcal{V}\text{-Cat}$ -category then $\mathcal{C} \times \mathcal{C}$ is also a $\mathcal{V}\text{-Cat}$ -category via the tensor operation \otimes in $\mathcal{V}\text{-Cat}$. We take advantage of this fact in the following definition.

► **Definition 19.** *A $\mathcal{V}\text{-Cat}$ -enriched autonomous category \mathcal{C} is an autonomous and $\mathcal{V}\text{-Cat}$ -category \mathcal{C} such that the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a $\mathcal{V}\text{-Cat}$ -functor and the adjunction $(-\otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}\text{-Cat}$ -adjunction.*

► **Example 20.** Recall that $\text{Pos} \simeq \mathcal{V}\text{-Cat}_{\text{sep}}$ when \mathcal{V} is the Boolean quantale. According to Theorem 17 the category Pos is autonomous. It follows by general results that the category is Pos -enriched [6]. It is also easy to see that its tensor is Pos -enriched and that the adjunction $(-\otimes X) \dashv (X \multimap -)$ is Pos -enriched. Therefore, Pos is an instance of Definition 19. Note also that $\text{Set} \simeq \mathcal{V}\text{-Cat}_{\text{sym, sep}}$ for \mathcal{V} the Boolean quantale and that Set is an instance of Definition 19.

Recall that $\text{Met} \simeq \mathcal{V}\text{-Cat}_{\text{sym, sep}}$ when \mathcal{V} is the metric quantale. Thus, the category Met is autonomous (Theorem 17) and Met -enriched [6]. It follows as well from routine calculations that its tensor is Met -enriched and that the adjunction $(-\otimes X) \dashv (X \multimap -)$ is Met -enriched. Therefore Met is an instance of Definition 19. An analogous reasoning tells that the category of ultrametric spaces (enriched over itself) is also an instance of Definition 19.

Finally, recall the interpretation of linear λ -terms on an autonomous category \mathcal{C} (Section 2) and assume that \mathcal{C} is $\mathcal{V}\text{-Cat}$ -enriched. Then we say that a \mathcal{V} -equation $\Gamma \triangleright v =_q w : \mathbb{A}$ is *satisfied* by this interpretation if $a(\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket, \llbracket \Gamma \triangleright w : \mathbb{A} \rrbracket) \geq q$ where $a : \mathcal{C}(\llbracket \Gamma \rrbracket, \llbracket \mathbb{A} \rrbracket) \times \mathcal{C}(\llbracket \Gamma \rrbracket, \llbracket \mathbb{A} \rrbracket) \rightarrow \mathcal{V}$ is the underlying function of the \mathcal{V} -category $\mathcal{C}(\llbracket \Gamma \rrbracket, \llbracket \mathbb{A} \rrbracket)$.

► **Theorem 21.** *The rules listed in Fig. 3 and Fig. 4 are sound for $\mathcal{V}\text{-Cat}$ -enriched autonomous categories \mathcal{C} . Specifically, if $\Gamma \triangleright v =_q w : \mathbb{A}$ results from the rules in Fig. 3 and Fig. 4 then $a(\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket, \llbracket \Gamma \triangleright w : \mathbb{A} \rrbracket) \geq q$.*

Proof. Let us focus first on the equations listed in Fig. 3. Recall that an equation $\Gamma \triangleright v = w : \mathbb{A}$ abbreviates the \mathcal{V} -equations $\Gamma \triangleright v =_{\top} w : \mathbb{A}$ and $\Gamma \triangleright w =_{\top} v : \mathbb{A}$. Moreover, we already know that the equations listed in Fig. 3 are sound for autonomous categories, specifically if $v = w$ is an equation of Fig. 3 then $\llbracket v \rrbracket = \llbracket w \rrbracket$ in \mathcal{C} (Theorem 3). Thus, by the definition of a \mathcal{V} -category and by the assumption of \mathcal{V} being integral ($k = \top$) we obtain $a(\llbracket v \rrbracket, \llbracket w \rrbracket) \geq k = \top$ and $a(\llbracket w \rrbracket, \llbracket v \rrbracket) \geq k = \top$.

Let us now focus on the rules listed in Fig. 4. The first three rules follow from the definition of a \mathcal{V} -category and the transitivity property of \leq . Rule **(arch)** follows from the continuity of \mathcal{V} , specifically from the fact that q is the *least* upper bound of all elements r that are way-below q . Rule **(join)** follows from the definition of least upper bound. The remaining rules follow from the definition of the tensor functor \otimes in $\mathcal{V}\text{-Cat}$, the fact that \mathbf{C} is $\mathcal{V}\text{-Cat}$ -enriched, $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ is a $\mathcal{V}\text{-Cat}$ -functor, and the fact that $(- \otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}\text{-Cat}$ -adjunction. For example, for the sixth rule we reason as follows:

$$\begin{aligned}
& a(\llbracket f(v_1, \dots, v_n) \rrbracket, \llbracket f(w_1, \dots, w_n) \rrbracket) \\
&= a(\llbracket f \rrbracket \cdot (\llbracket v_1 \rrbracket \otimes \dots \otimes \llbracket v_n \rrbracket) \cdot \mathbf{sp}_{\Gamma_1; \dots; \Gamma_n} \cdot \mathbf{sh}_E, \llbracket f \rrbracket \cdot (\llbracket w_1 \rrbracket \otimes \dots \otimes \llbracket w_n \rrbracket) \cdot \mathbf{sp}_{\Gamma_1; \dots; \Gamma_n} \cdot \mathbf{sh}_E) \\
&\geq a(\llbracket f \rrbracket \cdot (\llbracket v_1 \rrbracket \otimes \dots \otimes \llbracket v_n \rrbracket), \llbracket f \rrbracket \cdot (\llbracket w_1 \rrbracket \otimes \dots \otimes \llbracket w_n \rrbracket)) \\
&\geq a(\llbracket v_1 \rrbracket \otimes \dots \otimes \llbracket v_n \rrbracket, \llbracket w_1 \rrbracket \otimes \dots \otimes \llbracket w_n \rrbracket) \\
&\geq a(\llbracket v_1 \rrbracket, \llbracket w_1 \rrbracket) \otimes \dots \otimes a(\llbracket v_n \rrbracket, \llbracket w_n \rrbracket) \\
&\geq q_1 \otimes \dots \otimes q_n
\end{aligned}$$

where the second step follows from the fact that $\mathbf{sp}_{\Gamma_1; \dots; \Gamma_n} \cdot \mathbf{sh}_E$ is a morphism in \mathbf{C} and that \mathbf{C} is $\mathcal{V}\text{-Cat}$ -enriched. The third step follows from an analogous reasoning. The fourth step follows from the fact that $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ is a $\mathcal{V}\text{-Cat}$ -functor. The last step follows from the premise of the rule in question. As another example, the proof for the substitution rule proceeds similarly:

$$\begin{aligned}
& a(\llbracket v[v'/x] \rrbracket, \llbracket w[w'/x] \rrbracket) \\
&= a(\llbracket v \rrbracket \cdot \mathbf{jn}_{\Gamma, \mathbb{A}} \cdot (\mathbf{id} \otimes \llbracket v' \rrbracket) \cdot \mathbf{sp}_{\Gamma; \Delta}, \llbracket w \rrbracket \cdot \mathbf{jn}_{\Gamma, \mathbb{A}} \cdot (\mathbf{id} \otimes \llbracket w' \rrbracket) \cdot \mathbf{sp}_{\Gamma; \Delta}) \\
&\geq a(\llbracket v \rrbracket \cdot \mathbf{jn}_{\Gamma, \mathbb{A}} \cdot (\mathbf{id} \otimes \llbracket v' \rrbracket), \llbracket w \rrbracket \cdot \mathbf{jn}_{\Gamma, \mathbb{A}} \cdot (\mathbf{id} \otimes \llbracket w' \rrbracket)) \\
&\geq a(\mathbf{id} \otimes \llbracket v' \rrbracket, \mathbf{id} \otimes \llbracket w' \rrbracket) \otimes a(\llbracket v \rrbracket \cdot \mathbf{jn}_{\Gamma, \mathbb{A}}, \llbracket w \rrbracket \cdot \mathbf{jn}_{\Gamma, \mathbb{A}}) \\
&\geq a(\mathbf{id} \otimes \llbracket v' \rrbracket, \mathbf{id} \otimes \llbracket w' \rrbracket) \otimes a(\llbracket v \rrbracket, \llbracket w \rrbracket) \\
&\geq a(\mathbf{id}, \mathbf{id}) \otimes a(\llbracket v' \rrbracket, \llbracket w' \rrbracket) \otimes a(\llbracket v \rrbracket, \llbracket w \rrbracket) \\
&= a(\llbracket v' \rrbracket, \llbracket w' \rrbracket) \otimes a(\llbracket v \rrbracket, \llbracket w \rrbracket) \\
&\geq q \otimes r
\end{aligned}$$

The proof for the rule concerning (\multimap_i) additionally requires the following two facts: if a \mathcal{V} -functor $f : (X, a) \rightarrow (Y, b)$ is an isomorphism then $a(x, x') = b(f(x), f(x'))$ for all $x, x' \in X$. For a context Γ , the morphism $\mathbf{jn}_{\Gamma; x: \mathbb{A}} : \llbracket \Gamma \rrbracket \otimes \llbracket \mathbb{A} \rrbracket \rightarrow \llbracket \Gamma, x : \mathbb{A} \rrbracket$ is an isomorphism in \mathbf{C} . The proof for the rule concerning the permutation of variables (exchange) also makes use of the fact that $\llbracket \Delta \rrbracket \rightarrow \llbracket \Gamma \rrbracket$ is an isomorphism. \blacktriangleleft

3.3 Soundness, completeness, and internal language

In this subsection we establish a formal correspondence between linear $\mathcal{V}\lambda$ -theories and $\mathcal{V}\text{-Cat}$ -enriched autonomous categories, via soundness, completeness, and internal language theorems. A key construct in this correspondence is the quotienting of a \mathcal{V} -category into a *separated* \mathcal{V} -category: we will use it to identify linear λ -terms when generating a syntactic category (from a linear $\mathcal{V}\lambda$ -theory) that satisfies the axioms of autonomous categories. This naturally leads to the following notion of a model for linear $\mathcal{V}\lambda$ -theories.

► **Definition 22** (Models of linear $\mathcal{V}\lambda$ -theories). *Consider a linear $\mathcal{V}\lambda$ -theory $((G, \Sigma), Ax)$ and a $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched autonomous category \mathbf{C} . Suppose that for each $X \in G$ we have an interpretation $\llbracket X \rrbracket$ as a \mathbf{C} -object and analogously for the operation symbols. This interpretation structure is a model of the theory if all axioms in Ax are satisfied by the interpretation.*

Another thing that we need to take into account is the size of categories. In Section 2 we did not assume that autonomous categories should be locally small. In particular linear λ -theories are able to generate non-(locally small) categories. Now we need to be stricter because $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched autonomous categories are always locally small (recall the definition of $\mathcal{V}\text{-Cat}_{\text{sep}}$). Thus for two types \mathbb{A} and \mathbb{B} of a $\mathcal{V}\lambda$ -theory T , consider the class $\text{Values}(\mathbb{A}, \mathbb{B})$ of values v such that $x : \mathbb{A} \triangleright v : \mathbb{B}$. We equip $\text{Values}(\mathbb{A}, \mathbb{B})$ with the function $a : \text{Values}(\mathbb{A}, \mathbb{B}) \times \text{Values}(\mathbb{A}, \mathbb{B}) \rightarrow \mathcal{V}$ defined by,

$$a(v, w) = \bigvee \{q \mid v =_q w \text{ is a theorem of } T\}$$

It is easy to see that $(\text{Values}(\mathbb{A}, \mathbb{B}), a)$ is a (possibly large) \mathcal{V} -category. We then quotient this \mathcal{V} -category into a *separated* \mathcal{V} -category which we suggestively denote by $\mathbf{C}(\mathbb{A}, \mathbb{B})$ (as detailed in the proof of the next theorem, $\mathbf{C}(\mathbb{A}, \mathbb{B})$ will serve as a hom-object of a syntactic category \mathbf{C} generated from a linear $\mathcal{V}\lambda$ -theory). Following the nomenclature of [26], we call T *varietal* if $\mathbf{C}(\mathbb{A}, \mathbb{B})$ is a *small* \mathcal{V} -category. In the rest of the paper we will only work with varietal theories and locally small categories.

► **Theorem 23 (Soundness & Completeness).** *Consider a varietal $\mathcal{V}\lambda$ -theory. A \mathcal{V} -equation-in-context $\Gamma \triangleright v =_q w : \mathbb{A}$ is a theorem iff it holds in all models of the theory.*

Proof sketch. Soundness follows by induction over the rules that define the class $Th(Ax)$ (Definition 12) and by Theorem 21. For completeness, we use a strategy similar to the proof of Theorem 6, and take advantage of the quotienting of a \mathcal{V} -category into a separated \mathcal{V} -category. Recall that we assume that the theory is *varietal* and therefore can safely take $\mathbf{C}(\mathbb{A}, \mathbb{B})$ to be a small \mathcal{V} -category. Note that the quotienting process identifies all terms $x : \mathbb{A} \triangleright v : \mathbb{B}$ and $x : \mathbb{A} \triangleright w : \mathbb{B}$ such that $v =_{\top} w$ and $w =_{\top} v$. Such a relation contains the equations-in-context from Fig. 3 and moreover it is straightforward to show that it is compatible with the term formation rules of linear λ -calculus (Fig. 1). So, analogously to Theorem 6 we obtain an autonomous category \mathbf{C} whose objects are the types of the language and whose hom-sets are the underlying sets of the \mathcal{V} -categories $\mathbf{C}(\mathbb{A}, \mathbb{B})$.

Our next step is to show that the category \mathbf{C} has a $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched autonomous structure. We start by showing that the composition map $\mathbf{C}(\mathbb{A}, \mathbb{B}) \otimes \mathbf{C}(\mathbb{B}, \mathbb{C}) \rightarrow \mathbf{C}(\mathbb{A}, \mathbb{C})$ is a \mathcal{V} -functor:

$$\begin{aligned} a([v'], [v]), ([w'], [w]) &= a([v], [w]) \otimes a([v'], [w']) \\ &= a(v, w) \otimes a(v', w') \\ &= \bigvee \{q \mid v =_q w\} \otimes \bigvee \{r \mid v' =_r w'\} \\ &= \bigvee \{q \otimes r \mid v =_q w, v' =_r w'\} \\ &\leq \bigvee \{q \mid v[v'/x] =_q w[w'/x]\} && (A \subseteq B \Rightarrow \bigvee A \leq \bigvee B) \\ &= a(v[v'/x], w[w'/x]) \\ &= a([v[v'/x]], [w[w'/x]]) \\ &= a([v] \cdot [v'], [w] \cdot [w']) \end{aligned}$$

The fact that $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ is a $\mathcal{V}\text{-Cat}$ -functor follows by an analogous reasoning. Next, we need to show that $(-\otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}\text{-Cat}$ -adjunction. It is straightforward to show that both functors are $\mathcal{V}\text{-Cat}$ -functors, and from a similar reasoning it follows that the isomorphism $\mathbf{C}(\mathbb{B}, \mathbb{A} \multimap \mathbb{C}) \simeq \mathbf{C}(\mathbb{B} \otimes \mathbb{A}, \mathbb{C})$ is a \mathcal{V} -isomorphism.

The final step is to show that if an equation $\Gamma \triangleright v =_q w : \mathbb{A}$ with $q \in B$ is satisfied by \mathbf{C} then it is a theorem of the linear $\mathcal{V}\lambda$ -theory. By assumption $a([v], [w]) = a(v, w) = \bigvee \{r \mid v =_r w\} \geq q$. It follows from the definition of the way-below relation that for all $x \in B$ with

$x \ll q$ there exists a *finite* set $A \subseteq \{r \mid v =_r w\}$ such that $x \leq \bigvee A$. Then by an application of rule **(join)** in Fig. 4 we obtain $v =_{\bigvee A} w$, and consequently rule **(weak)** in Fig. 4 provides $v =_x w$ for all $x \ll q$. Finally, by an application of rule **(arch)** in Fig. 4 we deduce that $v =_q w$ is part of the theory. \blacktriangleleft

Next we establish results that will be key in the proof of the internal language theorem. Let $\text{Syn}(T)$ be syntactic category of a linear $\mathcal{V}\lambda$ -theory T , as described in Theorem 23.

► **Theorem 24.** *Consider a linear $\mathcal{V}\lambda$ -theory T and a model of T on a $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched autonomous category \mathcal{C} . The model induces a $\mathcal{V}\text{-Cat}_{\text{sep}}$ -functor $\text{Syn}(T) \rightarrow \mathcal{C}$ that (strictly) preserves the autonomous structure of $\text{Syn}(T)$.*

Proof. Consider a model of T over \mathcal{C} . Let a denote the underlying function of the hom- $(\mathcal{V}\text{-categories})$ in $\text{Syn}(T)$ and b the underlying function of the hom- $(\mathcal{V}\text{-categories})$ in \mathcal{C} . Then note that if $[v] = [w]$ then, by completeness, the equations $v =_{\top} w$ and $w =_{\top} v$ are theorems, which means that $\llbracket v \rrbracket = \llbracket w \rrbracket$ by the definition of a model and *separability*. This allows us to define a mapping $F : \text{Syn}(T) \rightarrow \mathcal{C}$ that sends each type \mathbb{A} to $\llbracket \mathbb{A} \rrbracket$ and each morphism $[v]$ to $\llbracket v \rrbracket$. The fact that this mapping is an autonomous functor follows from an analogous reasoning to the one used in the proof of Theorem 7. We now need to show that this functor is $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched. Recall that $a([v], [w]) = \bigvee \{q \mid v =_q w\}$ and observe that for every $v =_q w$ in the previous quantification we have $b(\llbracket v \rrbracket, \llbracket w \rrbracket) \geq q$ (by the definition of a model), which establishes, by the definition of a least upper bound, $a([v], [w]) = \bigvee \{q \mid v =_q w\} \leq b(\llbracket v \rrbracket, \llbracket w \rrbracket)$. \blacktriangleleft

Consider now a $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched autonomous category \mathcal{C} . It induces a linear $\mathcal{V}\lambda$ -theory $\text{Lang}(\mathcal{C})$ whose ground types and operations symbols are defined as in the case of linear λ -theories (recall Section 2). The axioms of $\text{Lang}(\mathcal{C})$ are all the \mathcal{V} -equations-in-context that are satisfied by the obvious interpretation on \mathcal{C} .

► **Theorem 25.** *The linear $\mathcal{V}\lambda$ -theory $\text{Lang}(\mathcal{C})$ is varietal.*

In conjunction with the proof of Theorem 23, a consequence of this last theorem is that $\text{Syn}(\text{Lang}(\mathcal{C}))$ is a $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched category. Then we state,

► **Theorem 26 (Internal language).** *For every $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched autonomous category \mathcal{C} there exists a $\mathcal{V}\text{-Cat}_{\text{sep}}$ -equivalence of categories $\text{Syn}(\text{Lang}(\mathcal{C})) \simeq \mathcal{C}$.*

Proof. Let a denote the underlying function of the hom- $(\mathcal{V}\text{-categories})$ in $\text{Syn}(\text{Lang}(\mathcal{C}))$ and b the underlying function of the hom- $(\mathcal{V}\text{-categories})$ in \mathcal{C} . We have, by construction, a model of $\text{Lang}(\mathcal{C})$ on \mathcal{C} which acts as the identity in the interpretation of ground types and operation symbols. We can then appeal to Theorem 24 to establish a $\mathcal{V}\text{-Cat}_{\text{sep}}$ -functor $\text{Syn}(\text{Lang}(\mathcal{C})) \rightarrow \mathcal{C}$. Next, the functor working on the inverse direction behaves as the identity on objects and sends a morphism f into $\llbracket f(x) \rrbracket$. Let us show that it is $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched. First, observe that if $q \ll b(f, g)$ in \mathcal{C} and $q \in B$ then $f(x) =_q g(x)$ is a theorem of $\text{Lang}(\mathcal{C})$, due to the fact that $\ll \llcorner$ entails \leq and by the definition of $\text{Lang}(\mathcal{C})$. Using the definition of a basis, we thus obtain $b(f, g) = \bigvee \{q \in B \mid q \ll b(f, g)\} \leq \bigvee \{q \in B \mid f(x) =_q g(x)\} = a(\llbracket f(x) \rrbracket, \llbracket g(x) \rrbracket)$. The equivalence of categories is then shown as in the proof of Theorem 8. \blacktriangleleft

All the results in this section can be extended straightforwardly to the case of *symmetric* linear $\mathcal{V}\lambda$ -theories and $\mathcal{V}\text{-Cat}_{\text{sym, sep}}$ -enriched autonomous categories.

4

 Examples of linear $\mathcal{V}\lambda$ -theories and their models

► **Example 27** (Wait calls). We now return to the example of wait calls and the corresponding metric axioms (1) sketched in the Introduction. Let us build a model over \mathbf{Met} for this theory: fix a metric space A , interpret the ground type X as $\mathbb{N} \otimes A$ and the operation symbol $\mathbf{wait}_n : X \rightarrow X$ as the non-expansive map, $\llbracket \mathbf{wait}_n \rrbracket : \mathbb{N} \otimes A \rightarrow \mathbb{N} \otimes A$, $(i, a) \mapsto (i + n, a)$. Since we already know that \mathbf{Met} is enriched over itself (recall Definition 19 and Example 20) we only need to show that the axioms in (1) are satisfied by the proposed interpretation. This can be shown via a few routine calculations.

Now, it may be the case that is unnecessary to know the *distance* between the execution time of two programs – instead it suffices to know whether a program finishes its execution *before* another one. This leads us to linear $\mathcal{V}\lambda$ -theories where \mathcal{V} is the Boolean quantale. We call such theories *linear ordered λ -theories*. Recall the language from the Introduction with a single ground type X and the signature of wait calls $\Sigma = \{\mathbf{wait}_n : X \rightarrow X \mid n \in \mathbb{N}\}$. Then we adapt the metric axioms (1) to the case of the Boolean quantale by considering instead:

$$\mathbf{wait}_0(x) = x \quad \mathbf{wait}_n(\mathbf{wait}_m(x)) = \mathbf{wait}_{n+m}(x) \quad \frac{n \leq m}{\mathbf{wait}_n(x) \leq \mathbf{wait}_m(x)}$$

where a classical equation $v = w$ is shorthand for $v \leq w$ (*i.e.* $v =_1 w$) and $w \leq v$ (*i.e.* $w =_1 v$). In the resulting theory we can consider for instance (and omitting types for simplicity) the λ -term that defines the composition of two functions $\lambda f. \lambda g. g (f x)$, which we denote by v , and show that $v (\lambda x. \mathbf{wait}_1(x)) \leq v (\lambda x. \mathbf{wait}_1(\mathbf{wait}_1(x)))$. This inequation between higher-order programs arises from the argument $\lambda x. \mathbf{wait}_1(\mathbf{wait}_1(x))$ being costlier than the argument $\lambda x. \mathbf{wait}_1(x)$ – specifically, the former will invoke one more wait call (\mathbf{wait}_1) than the latter. Moreover, the inequation entails that for every argument g the execution time of computation $v (\lambda x. \mathbf{wait}_1(x)) g$ will always be smaller than that of computation $v (\lambda x. \mathbf{wait}_1(\mathbf{wait}_1(x))) g$ since it invokes one more wait call. Thus in general the inequation tells that costlier programs fed as input to v will result in longer execution times when performing the corresponding computation. In order to build a model for the ordered theory of wait calls, we consider a poset A and define a model over \mathbf{Pos} by sending X into $\mathbb{N} \otimes A$ and $\mathbf{wait}_n : X \rightarrow X$ to the monotone map $\llbracket \mathbf{wait}_n \rrbracket : \mathbb{N} \otimes A \rightarrow \mathbb{N} \otimes A$, $(i, a) \mapsto (i + n, a)$. Since we already know that \mathbf{Pos} is enriched over itself (recall Definition 19 and Example 20) we only need to show that the ordered axioms are satisfied by the proposed interpretation. But again, this can be shown via a few routine calculations.

► **Example 28** (Probabilistic programs). We consider ground types $\mathbf{Real}, \mathbf{Real}^+, \mathbf{unit}$ and a signature consisting of $\{r : \mathbb{I} \rightarrow \mathbf{Real} \mid r \in \mathbb{Q}\} \cup \{r^+ : \mathbb{I} \rightarrow \mathbf{Real}^+ \mid r \in \mathbb{Q}_{\geq 0}\} \cup \{r^u : \mathbb{I} \rightarrow \mathbf{unit} \mid r \in [0, 1] \cap \mathbb{Q}\}$, an operation $+$ of type $\mathbf{Real}, \mathbf{Real} \rightarrow \mathbf{Real}$, and *sampling* functions $\mathbf{bernoulli} : \mathbf{Real}, \mathbf{Real}, \mathbf{unit} \rightarrow \mathbf{Real}$ and $\mathbf{normal} : \mathbf{Real}, \mathbf{Real}^+ \rightarrow \mathbf{Real}$. Whenever no ambiguities arise, we drop the superscripts in r^u and r^+ . Operationally, $\mathbf{bernoulli}(x, y, p)$ generates a sample from the Bernoulli distribution with parameter p on the set $\{x, y\}$, whilst $\mathbf{normal}(x, y)$ generates a normal deviate with mean x and standard deviation y . We then postulate the metric axiom,

$$\frac{p, q \in [0, 1] \cap \mathbb{Q}}{\mathbf{bernoulli}(x_1, x_2, p(*)) =_{|p-q|} \mathbf{bernoulli}(x_1, x_2, q(*))} \quad (3)$$

We interpret the resulting linear metric λ -theory in the category \mathbf{Ban} of Banach spaces and short operators, *i.e.* the semantics of [9, 21] without the order structure needed to interpret **while** loops. This is the usual representation of Markov chains/kernels as matrices/operators.

► **Theorem 29.** *The category \mathbf{Ban} is a Met-enriched autonomous category, and thus an instance of Definition 19.*

In particular, \mathbf{Ban} forms a model for the theory of our small probabilistic language via the following interpretation. We define $\llbracket \mathbf{Real} \rrbracket = \mathcal{M}\mathbb{R}$, the Banach space of finite Borel measures on \mathbb{R} equipped with the total variation norm, and similarly $\llbracket \mathbf{Real}^+ \rrbracket = \mathcal{M}\mathbb{R}^+$ and $\llbracket \mathbf{unit} \rrbracket = \mathcal{M}[0, 1]$. We have $\llbracket \mathbf{I} \rrbracket = \mathbb{R} \ni 1$, and for every $r \in \mathbb{Q}$ we put $\llbracket r \rrbracket : \mathbb{R} \rightarrow \mathcal{M}\mathbb{R}$, $x \mapsto x\delta_r$, where δ_r is the Dirac delta over r ; thus $\llbracket r \rrbracket(1) = \delta_r$. We define an analogous interpretation for the operation symbols r^+ and r^u . For $\mu, \nu \in \mathcal{M}\mathbb{R}$ we define $\llbracket + \rrbracket(\mu \otimes \nu) \triangleq +_*(\mu \otimes \nu)$ the pushforward under $+$ of the product measure $\mu \otimes \nu$ (seen as an element of $\mathcal{M}\mathbb{R} \otimes \mathcal{M}\mathbb{R}$, see [9]). For $\mu, \nu, \xi \in \mathcal{M}\mathbb{R}$ we define $\llbracket \mathbf{bernoulli} \rrbracket(\mu \otimes \nu \otimes \xi) \triangleq \mathbf{bern}_*(\mu \otimes \nu \otimes \xi)$, the pushforward of the product measure $\mu \otimes \nu \otimes \xi$ under the Markov kernel $\mathbf{bern} : \mathbb{R}^3 \rightarrow \mathbb{R}$, $(u, v, p) \mapsto p\delta_u + (1-p)\delta_v$, and similarly for $\llbracket \mathbf{normal} \rrbracket$ (see [9] for the definition of pushforward by a Markov kernel).

This interpretation is sound (a proof is given in [10]) because the norm on $\mathcal{M}\mathbb{R}$ is the total variation norm, and the metric axiom (3) describes the total variation distance between the corresponding Bernoulli distributions. Consider now the following λ -terms (where we abbreviate the constants $0(*), 1(*), p(*), q(*)$ to $0, 1, p, q$, respectively),

$$\begin{aligned} \mathbf{walk1} &\triangleq \lambda x : \mathbf{Real.bernoulli}(0, x + \mathbf{normal}(0, 1), p) \\ \mathbf{walk2} &\triangleq \lambda x : \mathbf{Real.bernoulli}(0, x + \mathbf{normal}(0, 1), q), \quad p, q \in [0, 1] \cap \mathbb{Q}. \end{aligned}$$

As the names suggest, these two terms of type $\mathbf{Real} \multimap \mathbf{Real}$ are denoted by random walks on \mathbb{R} . At each call, $\mathbf{walk1}$ (resp. $\mathbf{walk2}$) performs a jump drawn randomly from a standard normal distribution, or is forced to return to the origin with probability p (resp. q). These are non-standard random walks whose semantics are concretely given by complicated operators $\mathcal{M}\mathbb{R} \rightarrow \mathcal{M}\mathbb{R}$, but the simple quantitative equational system of Fig. 4 and the axiom (3) allow us to easily derive $\mathbf{walk1} =_{|p-q|} \mathbf{walk2}$ without having to compute the semantics of these terms. In other words, the soundness of (3) is enough to tightly bound the distance between two non-trivial random walks represented as higher-order terms in a probabilistic programming language. Furthermore, the tensor in the λ -calculus allows us to easily scale up this reasoning to random walks in higher dimensions such as $\mathbf{walk1} \otimes \mathbf{walk2}$ on \mathbb{R}^2 .

5 Conclusions and future work

We introduced the notion of a \mathcal{V} -equation which generalises the well-established notions of equation, inequation [23, 2], and metric equation [30, 31]. We then presented a sound and complete \mathcal{V} -equational system for linear λ -calculus, illustrated with different examples of programs containing real-time and probabilistic behaviour.

Functorial connection to previous work. As a concluding note, let us introduce a simple yet instructive functorial connection between (1) the categorical semantics of linear λ -calculus with the \mathcal{V} -equational system, (2) the categorical semantics of linear λ -calculus with the equational system of Section 2, and (3) the algebraic semantics of the exponential free, multiplicative fragment of linear logic. First we need to recall some well-known facts. As detailed before, typical categorical models of linear λ -calculus and its equational system are locally small autonomous categories. The latter form a quasicategory \mathbf{Aut} whose morphisms are autonomous functors. The usual algebraic models of the exponential free, multiplicative fragment of linear logic are the so-called *lineales* [11]. In a nutshell, a lineale is a poset (X, \leq) paired with a commutative, monoid operation $\otimes : X \times X \rightarrow X$ that satisfies certain

conditions. Lineales are almost quantales: the only difference is that they do not require X to be cocomplete. The key idea in algebraic semantics is that the order \leq in the lineale encodes the logic's entailment relation. A functorial connection between autonomous categories and lineales (*i.e.* between (2) and (3)) is stated in [11] and is based on the following two observations. First, (possibly large) lineales can be seen as *thin* autonomous categories, *i.e.* as elements of the enriched quasicategory $\{0, 1\}\text{-Aut}$. Second, the inclusion $\{0, 1\}\text{-Aut} \hookrightarrow \text{Aut}$ has a left adjoint which *collapses all morphisms* of a given autonomous category \mathbf{C} (intuitively, it eliminates the ability of \mathbf{C} to differentiate different terms between two types). This provides an adjoint situation between (2) and (3). We can now expand this connection to our categorical semantics of linear λ -calculus and corresponding \mathcal{V} -equational system (*i.e.* (1)) in the following way. The forgetful functor $\mathcal{V}\text{-Cat} \rightarrow \text{Set}$ has a left adjoint $D : \text{Set} \rightarrow \mathcal{V}\text{-Cat}$ which sends a set X to $DX = (X, d)$, with $d(x_1, x_2) = k$ if $x_1 = x_2$ and $d(x_1, x_2) = \perp$ otherwise. This left adjoint is strong monoidal, specifically we have $D(X_1 \times X_2) = DX_1 \otimes DX_2$ and $\mathbb{I} = (1, (*, *) \mapsto k) = D1$. This gives rise to the functors,

$$(\mathcal{V}\text{-Cat})\text{-Aut} \xleftarrow[\perp]{\hat{D}} \text{Aut} \xrightarrow[\perp]{c} \{0, 1\}\text{-Aut}$$

where \hat{D} equips the hom-sets of an autonomous category with the corresponding discrete \mathcal{V} -category and c collapses all morphisms of an autonomous category as described earlier. The right adjoint of \hat{D} forgets the \mathcal{V} -categorical structure between terms (*i.e.* morphisms) and the right adjoint of c is the inclusion functor mentioned earlier. Note that \hat{D} restricts to $(\mathcal{V}\text{-Cat}_{\text{sep}})\text{-Aut}$ and $(\mathcal{V}\text{-Cat}_{\text{sym, sep}})\text{-Aut}$, and thus we obtain a functorial connection between the categorical semantics of linear λ -calculus with the \mathcal{V} -equational system (*i.e.* (1)), (2), and (3). In essence, the connection formalises the fact that our categorical models admit a richer structure over terms (*i.e.* morphisms) than the categorical models of linear λ -calculus and its classical equational system. The latter in turn permits the existence of different terms between two types as opposed to the algebraic semantics of the exponential free, multiplicative fragment of linear logic. The connection also shows that models for (2) and (3) can be mapped into models of our categorical semantics by equipping the respective hom-sets with a trivial, discrete structure.

Future work. Recall that linear λ -calculus is at the root of different ramifications of λ -calculus that relax resource-based conditions in different ways. Currently, we are studying analogous ramifications of linear λ -calculus in the \mathcal{V} -equational setting, particularly affine and Cartesian versions. We are also studying the possibility of adding an exponential modality in order to obtain a *mixed linear-non-linear* calculus [3]. We also started to explore different definitions of a morphism between $\mathcal{V}\lambda$ -theories and respective categories. This is the basis to establish a categorical equivalence between a (quasi)category of $\mathcal{V}\lambda$ -theories and a (quasi)category of $\mathcal{V}\text{-Cat}_{\text{sep}}$ -enriched autonomous categories.

Next, our main examples of $\mathcal{V}\lambda$ -theories (see Section 4) used either the Boolean or the metric quantale. We would like to study linear $\mathcal{V}\lambda$ -theories whose underlying quantales are neither the Boolean nor the metric one, for example the ultrametric quantale which is (tacitly) used to interpret Nakano's guarded λ -calculus [5] and also to interpret a higher-order language for functional reactive programming [22]. Another interesting quantale is the Gödel one which is a basis for fuzzy logic [12] and whose \mathcal{V} -equations give rise to what we call fuzzy inequations.

Finally we plan to further explore the connections between our work and different results on metric universal algebra [30, 31, 36] and inequational universal algebra [23, 2, 36]. For example, an interesting connection is that the monad construction presented in [30] crucially relies on quotienting a pseudometric space into a metric space – this is a particular case of quotienting a \mathcal{V} -category into a separated \mathcal{V} -category (which we crucially use in our work).

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